Agenda for Friday, April 12, 2024

Reminders

- Office hours Fri 2PM in Locy 203
- ! Exam 1 on April 23 in discussion

Warm-up

How many subsets does $S = \{a, b\}$ have? What about $P = \{a, b, c\}$?
S has 4 subsets and P has 8 subsets.

Constructing new sets

Definition. Let X be a set. The power set of X is the set

$$\mathcal{P} \coloneqq \{A \mid A \subseteq X\}.$$

In other words, the power set is the set of all subsets of X.

If X is a finite set with cardinality n, then the power set $\mathcal{P}(X)$ has cardinality 2^n.

Definition. Let X and Y be sets. The Cartesian product of X and Y is the set

$$X \times Y \coloneqq \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

In other words, the Cartesian product is the set of all ordered pairs, where the first element is from X and the second element is from Y.

Definition. Let X_1, X_2,X_3, ..., X_n be a finite list of sets. The Cartesian product of $X_1, \ldots, X_n$ is the set

$$\prod_{i=1}^{n} X_i \coloneqq \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i \text{ for all } 1 \leq i \leq n\}.$$

In other words, the Cartesian product of n sets is the set of n-tuples where the i^{th} element comes the i^{th} set.

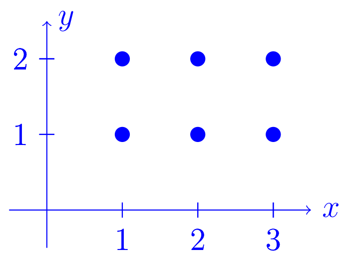
If $X_1 = X_2 = \cdots = X_n = X$, then the resulting product is also called the Cartesian power X^n

Name: ______________________________

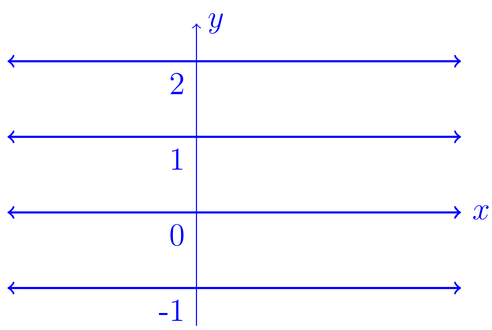
Practice

1. Each Cartesian product below is a subset of $\mathbb{R}^2$. Draw each set in the xy-plane.

   (a) $X \times Y$, where $X = \{1, 2, 3\}$ and $Y = \{1, 2\}$.

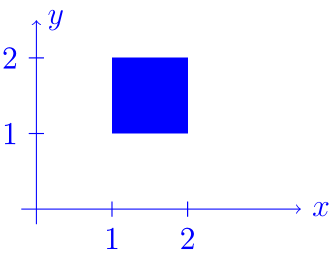
   (b) $X \times Y$, where $X = \mathbb{R}$ and $Y = \mathbb{Z}$.

   (c) $X \times Y$, where $X = \{1, 2, 3\}$ and $Y = \emptyset$

   This is the empty set

   (d) $X \times Y$, where $X = [1, 2]$ and $Y = [1, 2]$

2. List the elements of the set $\mathcal{P}(\{1,2\}) \times \mathcal{P}(\{3\})$. Be careful with brackets and parentheses!

   The 8 elements of the set are listed below.

$\mathcal{P}(\{1,2\}) \rightarrow$ / $\mathcal{P}(\{3\}) \downarrow$	$\emptyset$	$\{1\}$	$\{2\}$	$\{2,3\}$
$\emptyset$	$(\emptyset, \emptyset)$	$(\{1\}, \emptyset)$	$(\{2\}, \emptyset)$	$(\{2,3\}, \emptyset)$
$\{3\}$	$(\emptyset, \{3\})$	$(\{1\}, \{3\})$	$(\{2\}, \{3\})$	$(\{2,3\}, \{3\})$

3. What is the set described? Give a brief explanation.

   Note: Let $\mathbb{N} = \{1, 2, 3, \ldots\}$.

   (a) $\displaystyle\bigcup_{i \in \mathbb{N}} \mathbb{R} \times [i, i+1]$.

   This set is the subset of the Cartesian plane $\mathbb{R}^2$ of all points (x, y) such that $y \geq 1$. We could draw it by shading in all points of $\mathbb{R}^2$ on and above the horizontal line $y = 1$

   (b) $\displaystyle\bigcup_{X \in \mathcal{P}(\mathbb{N})} X$

   This set is $\mathbb{N}$.

4. If $J \neq \emptyset$ and $J \subseteq I$, does it follow that $\displaystyle\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$? What about $\displaystyle\bigcap_{\alpha \in J} A_\alpha \subseteq \bigcap_{\alpha \in I} A_\alpha$?

   It does follow that $\displaystyle\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$. For any element x in $\displaystyle\bigcup_{\alpha \in J} A_\alpha$, there exists $\beta \in J$ such that $x \in A_\beta$. Since $J \subseteq I$, we know $\beta \in I$ and A_β is part of the union on the right-hand side.

   The other statement about intersections is false. Since $J \subseteq I$, the intersection of sets of J is the intersection of *fewer* sets, which results in a *larger* final set.